MATH 155 - Chapter 9.1 - Sequences

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1. **Definition: (Infinite Sequence):** An **infinite sequence** is a function, a, defined on the set of positive integers such that for each positive integer n, there corresponds an real number $a(n)$. An infinite sequence is commonly denoted by

$$a(1),\ a(2),\ a(3),\ \cdots, a(n),\ \cdots$$

or commonly denoted by

$$a_1, a_2, a_3, \cdots, a_n, \cdots$$

The values are called the **terms** of the sequence. (a_1=first term, etc.) The abbreviation for the entire sequence is $\{a_n\}_1^\infty$ or $\{a_n\}$.

2. **Definition: (Convergence of a Sequence):** We say that the sequence $\{a_n\}$ **converges** to the real number L, or has the limit L, and write

$$\lim_{n\to\infty} a_n = L.$$

If the sequence does not converge, we say it **diverges**.

3. **Theorem: Limit Laws for Sequences:** If $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = k$, then

1. $\lim_{n\to\infty} (a_n \pm b_n) = L \pm K$
2. $\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n = cL.$
3. $\lim_{n\to\infty} (a_n b_n) = LK.$
4. $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{L}{K}, \qquad b_n \neq 0,\ \ K \neq 0.$

4. **Theorem: Squeeze Theorem for Sequences:** Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be sequences.

If $a_n \le c_n \le b_n$ for all n and $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} b_n$,
then $\lim_{n\to\infty} c_n = L.$

5. **Theorem: Absolute Value Theorem:**

If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

Note: The limit value must be 0. For other values, the theorem does not hold.

6. **Definition: (Monotonic Sequence):** A sequence $\{a_n\}$ is **monotonic** or **monotone increasing**, if its terms are non-decreasing

$$a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_n \leq \cdots.$$

A sequence $\{a_n\}$ is **monotone decreasing**, if its terms are non-increasing

$$a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq \cdots.$$

7. **Definition: (Bounded Sequence):**

1. A sequence $\{a_n\}$ is **bounded above** if there exits a real number M such that $a_n \leq M$ for all n. We call M=the upper bound for the sequence.

2. A sequence $\{a_n\}$ is **bounded below** if there exits a real number N such that $a_n \geq N$ for all n. We call N=the lower bound for the sequence.

3. A sequence $\{a_n\}$ is **bounded** if it is bounded above and below.

8. **Theorem: Bounded Monotonic Sequences:**

If $\{a_n\}$ is bounded and monotonic, then $\{a_n\}$ converges.